International Journal of Civil Engineering (IJCE)
ISSN(P): 2278-9987; ISSN(E): 2278-9995
Vol. 3, Issue 2, Mar 2014, 53-66

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

EFFECT OF PATCH LENGTH RATIO OF IN-PLANE LOADING ON THE POST BUCKLING BEHAVIOR OF RECTANGULAR THIN PLATE

JAWAD TALIB ABODI

Lecturer, Department of Civil Engineering, University of Kerbala, Karbala, Iraq

ABSTRACT

The present study investigates the problem of post buckling of thin steel plates subjected to in-plane patch compression loading. Finite difference method was used to treat the stability problems. The geometrically nonlinearity was considered. The present procedure is general and applicable to the buckling, post buckling and free vibration of thin rectangular plates. The influences of initial imperfection, thickness variation, plate aspect ratios, boundary conditions, and length of patch loading on the post buckling behavior are shown graphically. The plate was analyzed with different tapering ratios (t_a/t_o) (1.0, 1.25, 1.5, 1.75 and 2.0) so different patch length ratio (***S***) (0.0-0.3) were taken. A comparison with previous works is made. Finally, it is shown that the post buckling behavior very sensitive for some effects such as initial imperfection, tapering ratio, and patch length ratio.

KEYWORDS: Thin Plates, Tapered Plates, Large Displacement, Post Buckling Behavior, Patch Compression Loading, Finite Difference Method

INTRODUCTION

Thin plates are commonly used in most structures. Ships and marine components are examples of complex thin-walled structures that are composed of various plate elements characterized by different combinations of geometry and loading conditions. Post buckling behavior of thin plates with and without fault is important. Post buckling behavior of thin walled structures is a well-known phenomenon and because of occurring easily, it must be diagnosed accurately in security and safety considerations[9]. In the behavior of these plate structures under in-plane compression loads, a critical point exists where an infinitesimal increase in load can cause the plate surface to buckle. The load at this critical point defines the buckling strength of the plate. Increases in load beyond the critical load at the initiation of buckling increase the buckling deformations until collapse occurs. Thus, the load at collapse defines the post buckling or crippling strength of the plate. Thin plates are susceptible to different types of defects such as initial imperfections, aspect ratios, and boundary conditions, etc. On the other hand, a limited number of studies have been carried out to evaluate the influence of patch loading on the post buckling strength in the compressed plates although designers are always confronted with this issue. Such a problem is encountered in airframe where the action of the air loading on an aircraft wing develops an axial loading that gives a non-uniform compression that can lead to loss of stability. In addition, the aerodynamic heating of panels in supersonic aircraft can be approximated by non-uniform thermal stresses, as the temperature distribution is not uniform throughout the volume of the restrained plate. In civil engineering structures, engineers are often confronted with designs involving partial edge loading, such as the buckling of the web plate of a crane girder under the action of heavy wheel loads applied to the flanges. It is worth to point out that since constructional elements are frequently subjected to in-plane patch loading and often prone to buckling and post-buckling, it is important that further design data should be provided to deal with this important stability problem. If such an issue has so far received relatively little attention from researchers, the reason for this is undoubtedly due to the additional theoretical difficulties involved in obtaining rigorous solutions to the buckling of plate when subjected to non-uniform compression. Undeniably, the solution of this stability problem is mathematically difficult to obtain as the stress distribution throughout the plate varies considerably [4]. In 2010, **Ikhenazen**, ***et. al.*** investigate the problem of linear buckling of simply supported thin plates subjected to patch compression by using finite element method. The stability problem was treated by using the total energy and the plate was modelled by means of an eight nodes rectangular element and a reduction of variable strategy where applied to estimate the number of degrees of freedom leading to little or no loss in seeking solution accuracy. They concluded that a good accuracy of the minimal critical buckling load and a big saving in computer time have been obtained. **Abodi** (2012) investigate the problem of linear buckling of thin steel plates subjected to in-plane patch compression loading by using finite difference method. He studied influences of thickness variation, plate aspect ratios, and boundary conditions, and length of patch loading on the buckling load and shown graphically. The plate was analyzed with different tapering ratios (t_a/t_o) (1.0, 1.25, 1.5, 1.75 and 2.0), so different patch length ratio (S_p) (0.0-0.4) were taken. He concluded that the buckling load factor will increase with decreasing length of axial patch loading where the decreasing the length of axial patch loading to 0.4 will increase the buckling load factor by about 40% for plate with aspect ratio (a/b=1) and tapering ratio (t_a/t_o=1.0). In the present study, the post buckling of thin elastic plates non-uniformly compressed in one direction (see Figure (1)) is investigated using the finite difference method. This numerical analysis is performed with the FORTRAN90 program that was written by **Ammash**[3]. The aim of this paper is to show some representative elastic post buckling behavior of a simply supported plate under in-plane patch loading with constant and variable thickness. The influence of edge ratio and load breadth ratio on the post-buckling strength is investigated. The obtained numerical results are graphically summarized through an in-plane load with out of plane deformations, varying boundary condition, varying length of patch loading ratio and varying tapering ratio and some interesting conclusions are drawn.

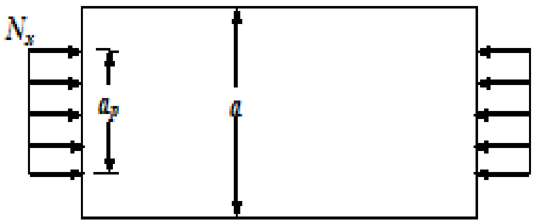

Figure 1: Rectangular Thin Plate under In-Plane Patch Loading

Basic Plate Relationships and Equations

For a homogeneous isotropic plate, the basic relationships for stresses and deformations of a plate element ($dx.dy.t$) may be summarized as follows:

In large deflection behavior the interaction between the flexural and the membrane actions is taken into account. In this case the deflection and the stresses vary in a nonlinear manner with the magnitude of the membrane load.

For large deflections of plate with constant thickness and with out-of-plane deformation (w_o) up to several times of the plate thickness (t), the basic differential equations are given as follows[3]:

$$D.\nabla^4 w = \left(q + \left(\frac{\partial^2 \Phi}{\partial y^2}\frac{\partial^2 (w+w_o)}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 (w+w_o)}{\partial y^2} - 2\frac{\partial^2 \Phi}{\partial x \partial y}\frac{\partial^2 (w+w_o)}{\partial x \partial y} \right)\right) \quad (1)$$

$$\nabla^4 \Phi = Et\left(\left(\frac{\partial^2 (w+w_o)}{\partial x \partial y} \right)^2 - \frac{\partial^2 (w+w_o)}{\partial x^2}\frac{\partial^2 (w+w_o)}{\partial y^2} \right) \quad (2)$$

Where

$$N_x = t.\frac{\partial^2 \Phi}{\partial y^2}; \quad N_y = t.\frac{\partial^2 \Phi}{\partial x^2}; \quad N_{xy} = -t.\frac{\partial^2 \Phi}{\partial x \partial y} \quad (3)$$

The basic differential equation for the membrane action is derived as follows:

Starting from the equation of compatibility of a thin plate element and expressing the strains and curvatures as functions of the stress resultants, the following equations are presented [3]:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{yx}}{\partial x \partial y} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + \frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2} = 0 \quad (4)$$

$$\varepsilon_x = \frac{(N_x - v.N_y)}{Et_x}$$

$$\varepsilon_y = \frac{(N_y - v.N_x)}{Et_x}$$

$$\gamma_{xy} = \frac{2(1+v)}{Et_x} N_{xy} \quad (5)$$

The required derivatives for strain in equations (5) are:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{1}{Et_x}\left(\frac{\partial^2 N_x}{\partial y^2} - v.\frac{\partial^2 N_y}{\partial x^2} \right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{1}{Et_x}\left(\frac{2c_t^2}{(1+c_t x)^2}(N_y - v.N_x) - \frac{2c_t}{(1+c_t x)}\left(\frac{\partial N_y}{\partial x} - v.\frac{\partial N_x}{\partial x} \right) + \left(\frac{\partial^2 N_y}{\partial x^2} - v.\frac{\partial^2 N_x}{\partial x^2} \right) \right)$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{2(1+v)}{Et_x}\left(\frac{\partial^2 N_{xy}}{\partial x \partial y} - \frac{c_t}{(1+c_t x)}.\frac{\partial N_{xy}}{\partial y} \right) \quad (6)$$

Substitution of these derivatives in equation (4) yields:

$$\frac{\partial^2 N_x}{\partial y^2}+\frac{\partial^2 N_y}{\partial x^2}-2.\frac{\partial^2 N_{xy}}{\partial x\partial y}-\frac{2c_t}{(1+c_t x)}\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_y}{\partial x}\right)$$
$$+\frac{2c_t^2}{(1+c_t x)^2}(N_y-v.N_x)-Et_x.\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2+Et_x\left(\frac{\partial^2 w}{\partial x^2}.\frac{\partial^2 w}{\partial y^2}\right)=0 \tag{7}$$

By expressing equation (7) as function of stress resultants by using equation (3), then:

$$\nabla^4\Phi-F\left(\frac{\partial^3\Phi}{\partial x^3}+\frac{\partial^3\Phi}{\partial x\partial y^2}\right)+Z\left(\frac{\partial^2\Phi}{\partial x^2}-v.\frac{\partial^2\Phi}{\partial y^2}\right)+Et_x\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)-$$
$$Et_x\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2=0 \tag{8}$$

Where

$$t_x=t_o(1+c_t x)$$

$$c_t=\frac{t_a-t_o}{at_o}$$

$$F=\frac{2c_t}{(1+c_t x)}$$

$$Z=\frac{2c_t^2}{(1+c_t x)^2}$$

By similar algebraic steps, it is possible to write the equilibrium equation in terms of w and Φ, thus

$$D_{(x)}.\nabla^4 w+2\frac{\partial^2 D_{(x)}}{\partial x}\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)+\frac{\partial^2 D_{(x)}}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right)=$$
$$\left[q+\frac{\partial^2\Phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\Phi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right] \tag{9}$$

These equations (8) and (9) may be considered as the basic (or governing) differential equations for a plate with variable thickness and subjected to transverse and in-plane compressive load, as shown in Figure (2).

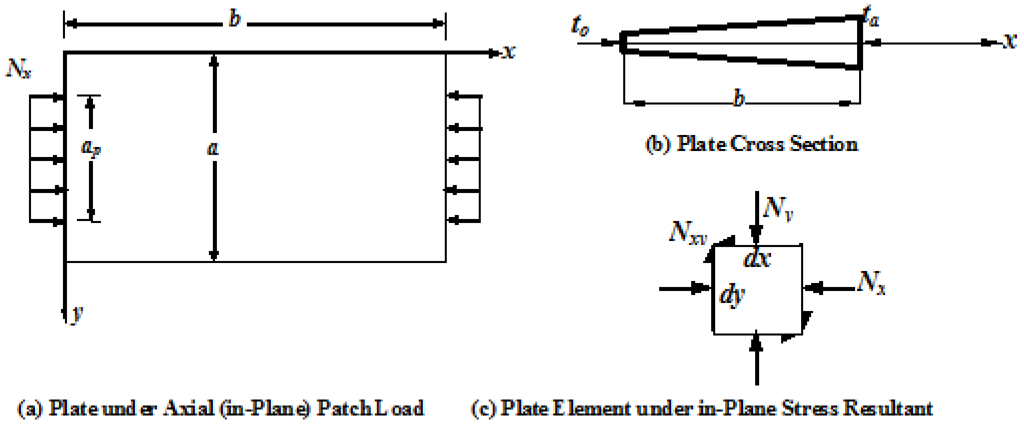

Figure 2: Axial (In-Plane) Load of Linearly Tapered Rectangular Plate

Boundary Conditions

For simply supported or hinged plate [3]:

Loaded edges (y=0 and y=a by N_x (per unit width))

a- Loaded edges (y=0 and y=a)

a-1- Flexural Boundary Conditions

At x = 0, b

w = 0, M_y = 0

where

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right) = 0$$

a-2- Membrane Boundary Conditions

at y = 0, a

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{-N_y}{t} \;;\quad \frac{\partial^2 \Phi}{\partial x \partial y} = 0$$

(No shearing stresses at edges)

b- Unloaded edges (x=0 and x=b)

b-1- Flexural Boundary Conditions

at x = 0, b

w = 0, M_x = 0

where

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right) = 0$$

a-2- Membrane Boundary Conditions

at $x = 0, b$

$$\frac{\partial^2 \Phi}{\partial y^2} = 0 \; ; \; \frac{\partial^2 \Phi}{\partial x \partial y} = 0$$

Solution Procedure

The finite difference procedure is employed here for the large deflection problems. The coupled equilibrium equations may be written by finite difference expressions as[(3)]:

$$[A]\{\Phi_1\} + [B]\{w_o\} = 0 \tag{10}$$

$$[C]\{w_1\} + [D]\{\Phi_1\}[M]\{w_o\} = 0 \tag{11}$$

in which

[A]: Stresses matrix.

[C]: Bending stiffness matrix.

$\{w_0\}$ **:** Initial displacement vector.

$\{w_1\}$ **:** New displacement vector.

$\{\Phi_1\}$ **:** Stress vector.

Outside (fictitious) nodes are needed for deflection (w_1) and also line integrated on along boundary is used for the stress function.

The following solution procedure is suggested in order to adequately determine the secondary buckling load and the secondary buckling mode: Definition of desired load level.

- As tshe out of plane displacement vector $\{w_0\}$ is not known; an initial displacement vector $\{w_0\}$ will be assumed likely a $\{0.0001\}$.
- Putting the assumed vector $\{w_0\}$ in Equation (8) to evaluate the stress vector $\{\Phi_1\}$.
- Putting the stress vector $\{\Phi_1\}$ from step (3) and displacement vector $\{w_0\}$ in Equation (9) to evaluate a new displacement vector $\{w_1\}$.

Steps (1-3) represent one cycle of the iterative procedure and the procedure is repeated until the desired convergence criterion is achieved. The whole procedure is repeated for a new load level[3].

NUMERICAL RESULTS

To study the effect of different parameters such as: initial imperfection ,thickness variation, plate aspect ratios, boundary conditions, and length of patch loading on the post-buckling behavior of rectangular thin plates, several plates are analyzed by using the finite difference method. Non-dimensional relationships between load and the out-of-plane displacements are given to show the post-buckling behavior of these plates under in-plane compressive patch load.

The accuracy of the results of the present program for the analysis of real panels was compared by Ammash[3] with the available experimental and numerical results obtained by Mirambell, *et al*[6] [1994] on simply supported panels. The properties of this specimen are shown in Figure (3). The numerical analysis of Mriambell, *et al* is based on the displacement formulation of the finite element method for the nonlinear analysis of general steel-shell structures.

Ammash was analyzed the plate based on the prescribed procedure and divided it into (24×12) divisions. The following comparison concern a plate model of thickness $\boldsymbol{t}$=3mm, $\boldsymbol{E}=2.1\times10^6$ kN/m^2, $\boldsymbol{a}$=600mm, $\boldsymbol{b}$=1200mm, $\boldsymbol{v}$=0.3, and the initial imperfection (w_o=**1.92 sin (2πx/b) sin (πy/a)** (in mm))

Figure (3) shows a comparison between the experimental and the numerical results for the out-of-plane displacements. The results obtained from Ammash's study were closer to the test results than to the finite element results obtained by Mirambell, *et al* [1994] because his study the differential equations directly but the finite element the uses approximate polynomial fields for elements. The load-deflection results are listed in Table (1).

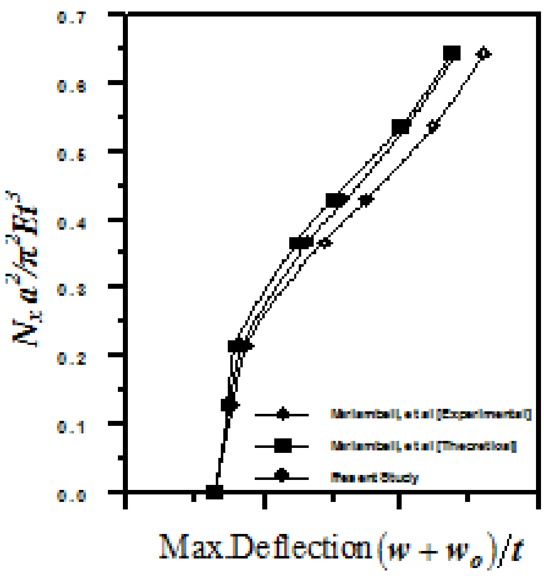

Figure 3: Post-Buckling Behavior of a Simply Supported Thin Rectangular Plate under Uniaxial Compression Load at *x*-Direction

Table 1: Comparison of Results with Experimental and Theoretical Studies

Maximum Deflection $((w+w_o)/t)$			*Load* $(N_x a^2/\pi^2 E t^3)$
Present Study	Theoretical Results()	Experimental Results()	
0.640	0.640	0.640	0
0.743	0.733	0.766	0.128
0.833	0.783	0.866	0.214
1.316	1.233	1.433	0.366
1.600	1.500	1.750	0.429
2.040	2.000	2.233	0.536
2.400	2.366	2.600	0.644

In 1975, Williams and **Walker**(13) derived explicit expression for the load-deflection relationship for simply supported uniformly square plate based on the perturbation approach. The results were presented for the plates with variety geometries, boundary constraints and in-plane loading conditions. The accuracy of these results was sufficient for engineering design purposes.

Figure (4) shows the load-out-of-plane displacements of a simply supported thin plate under compressive load (N_x (per unit width)). The following comparison concern a plate model of thickness t=0.01(m), E=2×10^6 kN/m^2 and v=0.3. The plate has an initial imperfection (w_o/t) of (0, 0.01, 0.1, and 0.5) of which the shape is considered to be sinusoidal ($w_o \sin\left(\pi x/a\right)\sin\left(\pi y/b\right)$) where w_o is the amplitude of the initial imperfection at the center of the plate. The results of the present study are compared with the results of Williams and Walker(13) study. Good agreement with these theoretical results is achieved.

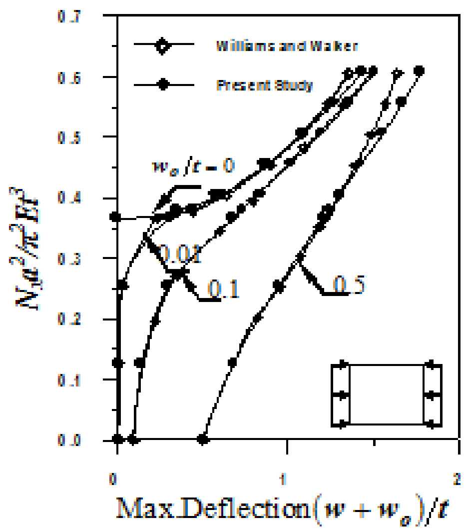

Figure 4: Post-Buckling Behavior of a Simply Supported Thin Square Plate under Uniaxial Compressive Load at *x*-Direction(3)

In all the presented cases, a finite difference method was used by considering the full plates with (14×14) mesh. The following geometry and material properties of steel plate are used in the analysis: (E=200 GPa; v=0.30, F_y=250 MPa,). The effect of patch length ratios on the post-buckling behavior is considered in the present study.

The values of patch length ratios (***S***=(***a***-***a***$_p$)/***a***) is taken to be (1.0, 0.9, 0.8, 0.7, 0.6, 0.5 and 0.4). The initial imperfection (***w***$_o$/***t***) is considered to be sinusoidal ($w_o \sin\left(\pi x/a\right)\sin\left(\pi y/b\right)$) where w_o is the amplitude of the initial imperfection at the center of the plate.

Figure (5) presents the load-deflection curve of a simply supported thin square perfect plate under uniaxial patch load in ***x***- direction with various ratios of patch length.

Figure (6) presents the load-deflection curve of a simply supported thin square imperfect plate under uniaxial patch load in *x*- direction with various ratios of patch length. The initial imperfection (***w***$_o$/***t***) is taken to be 0.1.

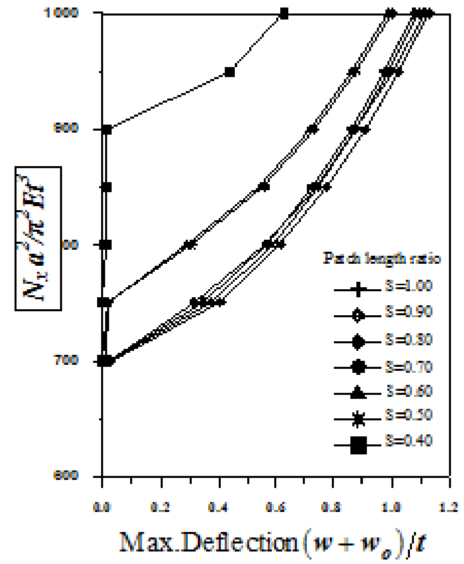

Figure 5: Post-Buckling Behavior of a Simply Supported Thin Square Plate under Uniaxial Compressive Load at *x*-Direction with Various Ratios of Patch Length

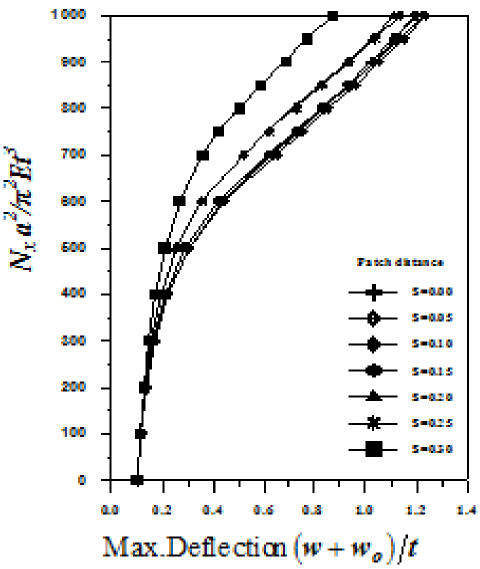

Figure 6: Post-Buckling Behavior of a Simply Supported Thin Square Plate under Uniaxial Compressive Load at *x*-Direction with Various Ratios of Patch Length

Figure (7) presents patch length ratio-maximum deflection curve of a simply supported thin square imperfect plate under uniaxial patch load in ***x***- direction. The initial imperfection (w_o/t) is taken to be (0.0, 0.10, 0.25, 0.50, 0.75 and 1.00) with sinusoidal curve. The value of in-plane of patch loading at ***x***-direction is taken to be 1000 kN/m and the slenderness ratio is (b/t=100) with aspect ratio (a/b=1.0).

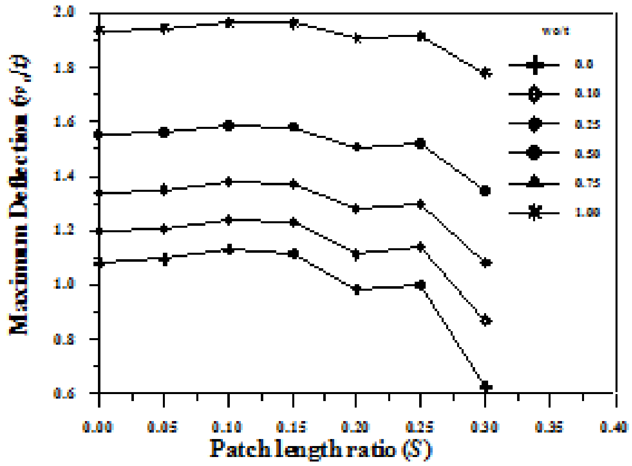

Figure 7: Post-Buckling Behavior of a Simply Supported Thin Square Plate under Uniaxial Compressive Load at *x*-Direction with Various Ratios of Patch Length

Figure (8) presents patch length ratio-maximum deflection curve of a clamped supported thin square imperfect plate under uniaxial patch load in ***x***- direction. The initial imperfection (w_o/t) is taken to be (0.0, 0.10, 0.25, 0.50, 0.75, and 1.00) with sinusoidal curve. The value of in-plane of patch loading at ***x***-direction is taken to be 2000 kN/m and the slenderness ratio is (b/t=100) with aspect ratio (a/b=1.0).

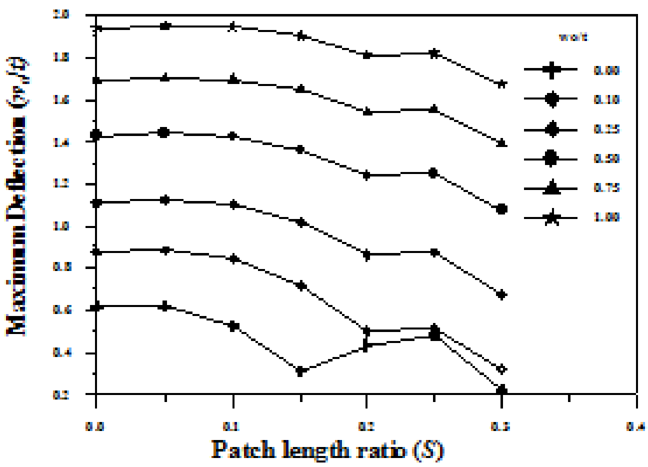

Figure 8: Post-Buckling Behavior of a Simply Supported Thin Square Plate under Uniaxial Compressive Load at *x*-Direction with Various Ratios of Patch Length

Figure (9) presents patch length ratio-maximum deflection curve of a simply supported thin square perfect plate under uniaxial patch load in ***x***- direction. Various values of aspect ratios were taken into account as (a/b=0.5, 1.0, 1.5, 2.0, 3.0 and 4.0). The initial imperfection (w_o/t) is taken to be zero. The value of in-plane of patch loading at ***x***-direction is taken to be 1000 kN/m and the slenderness ratio is (b/t=100).

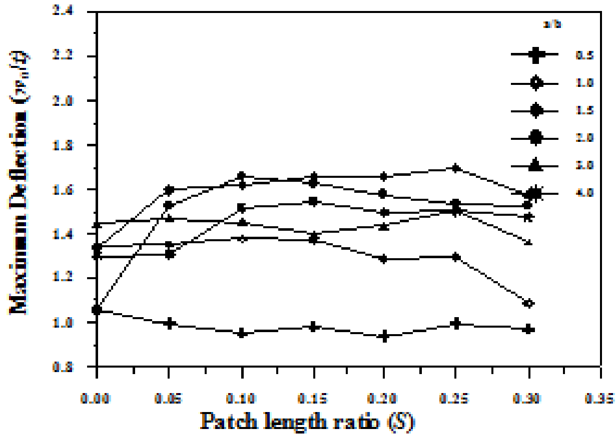

Figure 9: Post-Buckling Behavior of a Simply Supported Thin Square Plate under Uniaxial Compressive Load at *x*-Direction with Various Ratios of Patch Length

Figure (10) presents patch length ratio-maximum deflection curve of a simply supported thin square imperfect plate under uniaxial patch load in ***x***- direction. Various values of tapering ratios were taken into account as (t_a/t_o=1.0, 1.25, 1.5, 1.75, and 2.0). The initial imperfection (w_o/t) is taken to be 0.1 with sinusoidal curve. The value of in-plane of patch loading at ***x***-direction is taken to be 1000 kN/m and the slenderness ratio is (b/t=100) with aspect ratio (a/b=1.0).

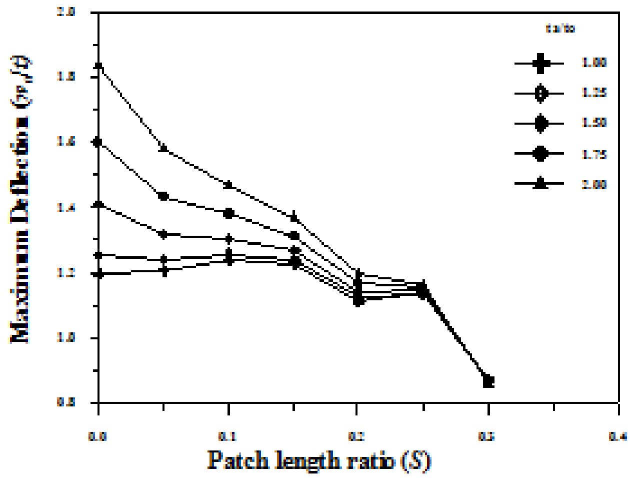

Figure 10: Post-Buckling Behavior of a Simply Supported Thin Square Plate under Uniaxial Compressive Load at *x*-Direction with Various Ratios of Patch Length

From these figures, can be noticed that:

- The post buckling behavior is effected by the patch length ratio where maximum deflection will decrease with increasing the patch length ratio.
- The percentage of decreasing of maximum deflection for simply supported plate under in-plane patch loading at ***x***-direction with initial imperfection equal to zero about (42.1%) while for same plate with initial imperfect equal to (1.0) about (8.1%) with patch length ratio (0.3).
- The percentage of decreasing of maximum deflection for clamped supported plate under in-plane patch loading at ***x***-direction with initial imperfection equal to zero about (62.1%) while for same plate with initial imperfect equal to (1.0) about (13.7%) with patch length ratio (0.3).

- The percentage of decreasing of maximum deflection for simply supported plate under in-plane patch loading at ***x***-direction with initial imperfection equal to zero with aspect ratio (***a***/***b***=1.0) about (19%) while for same plate with aspect ratio equal to (***a***/***b***=2.0) will increase about (45.3%) with patch length ratio (0.3).
- The percentage of decreasing of maximum deflection for simply supported plate under in-plane patch loading at ***x***-direction with initial imperfection equal to (0.1) with aspect ratio (***a***/***b***=1.0) about (27.1%) for plate with tapering ratio (1.0) while for plate with tapering ratio (2.0) about (53.3%).
- The percentage of decreasing of maximum deflection have same values for plate with tapering ratio (t_a/t_o=2.0), aspect ratio (***a***/***b***=1.0), slenderness ratio (***b***/***t***=100).

CONCLUSIONS

This paper was presented a general method of analyzing the post buckling behavior of rectangular plate with constant thickness or variable with initial curvatures. Finite difference method is very suitable for programming and sufficiently accurate as it tends to the exact solution when the node density is increased. The effect of initial imperfection, patch length ratio, aspect ratio, boundary condition, and tapering ratio on the post buckling behavior are considered. It is concluded that the post buckling behavior of thin plate is very sensitive to the magnitude of some effects such as patch length ratio, initial imperfection and magnitude of tapering ratio.

REFERENCES

1. Aalami,B., and Chapman, J.C., "Large Deflection Behavior of Orthotropic Plates under Transverse and In-plane Loads", Proc. ICE, London, England, Vol. 42, Mar., 1969, pp.347-382.
2. Abodi, J.T. "Elastic Buckling of Plates under In-Plane Patch Loading Using Finite Difference Method", Kerbala J., Vol. 10, No. 2, 2012, pp 142-150.
3. Ammash, h.k., "Post-buckling and Post-yielding Analysis of Imperfect Thin Plate by Finite Difference Method." M.Sc. thesis, University of Babylon, February 2003.
4. Ikhenazen,G., Saidani,M.mChelghoum, A. " Finite element analysis of linear plates buckling under in-plane patch loading ", Journal of Constructional Steel Research, Vol. 66, 2010, pp. 1112-1117.
5. Kopayashi, H., and Sonoda, K., "Buckling of Rectangular Plates with Tapered Thickness", ASCE, J. Struct. Eng., Vol. 116, No.5, May, 1990, pp. 1278-1289.
6. Mirambell, E., Costa, J., and Arnedo, A., "Analytical and Experimental Study on the Behavior of Steel Panels under Plane Compression", Proc. Int. Conf. On Steel Struct., CI-Premier Pte. Ltd, Jakarta, Indonesia, 1994, pp.205-212
7. Ohga, M., Shigematsu, T., and Kawagouchi, K., "Buckling Analysis of Thin-Walled Members with Variable Thickness", ASCE, J. Struct. Eng., Vol. 121, No. 6, June, 1995, pp. 919-924.
8. Pavlovic NM, Baker G. "Buckling of non-uniformly compressed plates" Theoret Appl Mecha. Transaction of the Yugoslav Society of Mechanics, vol. 9. 1983. pp. 91-103.

9. Salvadori, M.G., “Numerical Computation of Buckling Loads by Finite Differences”, ASCE, Trans., Dec., 1949, pp. 590-636.

10. Seifi, R., and Khoda-yari , N. “Experimental and Numerical Studies on Buckling of Cracked Thin Plates under Full and Partial Compression Edge Loading”, J. of Thin-walled Struct. Vol. 49, 2011, pp 1504-1516.

11. Timoshenko, S.P., and Woinowshy, S., “Theory of Plates and Shells”, 2nd Ed, McGraw-Hill Book Co., Inc., New York, 1959.

12. Timoshenko, S.P., and Gere, J.M., “Theory of Elastic Stability”, 2nd Ed, McGraw-Hill Book Co., Inc., New York, 1961.

13. Williams, D.G. and Walker, A. C., “Explicit Solution for The Design of Initially Deformed Plates Subject to Compression”, Proc. ICE, London, Vol. 59, Part 2, 1975, pp.763-787.